Solitary Rayleigh waves in the presence of surface nonlinearities

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The propagation of Rayleigh waves is investigated in a solid substrate of linear material covered by a film consisting of a material with large nonlinear elastic moduli. For this system, a nonlinear evolution equation is derived that may be regarded as a special case in a wider class of evolution equations with a specific type of nonlocal nonlinearity. Periodic pulse train solutions are computed. For a certain member of the class of nonlinear evolution equations, several families of solitary wave solutions and their associated periodic stationary wave solutions are derived analytically.

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INTRODUCTION

Recently, renewed interest (both experimental and theoretical) has developed in nonlinear surface acoustic waves. This is partly due to new experiments on the propagation of high-intensity surface acoustic pulses [1–3] and on wave form evolution of initially sinusoidal Rayleigh waves [4] and partly due to interesting and sometimes controversially discussed topics in the theory of nonlinear surface acoustic waves as the existence of stationary waves in the absence of linear dispersion [5–8] and shock formation or wave breaking of nondispersive nonlinear Rayleigh waves [9–14]. From a mathematical point of view, the nonlocality of the nonlinearity arising in the theoretical description of nonlinear Rayleigh waves [8,13,15–18] is an interesting phenomenon and its consequences are yet little explored.

It is well known that in the case of an isotropic elastic half space there exist two main types of surface waves: (1) Rayleigh waves, having sagittal polarization with two nonzero components of the displacement field (two-component field), which are coupled due to the boundary conditions at the surface, and (2) Love waves, having shear-horizontal polarization (one-component field). The latter only exist in the presence of a film of different material covering the substrate [19,20]. The density of energy in surface waves can be very large due to its localization in a narrow domain near the surface and so nonlinear effects can be expected to be significant for such types of waves. Nonlinear effects in connection with surface acoustic waves have been investigated in the past to a large extent, in the case of shear-horizontal waves mostly on the theoretical side focussing on envelope solitons [21] (for references to earlier work see Ref. [22]). Recently, a number of experimental studies have been carried out on nonlinear wave form evolution of Rayleigh waves in the presence of linear dispersion [23–25,4,2]. In some of these experiments [23,2], linear dispersion of the Rayleigh waves was generated by covering the substrate with a film made of a different material.

In these earlier investigations, the dominant nonlinearity has been that of the substrate. As the film thickness has mostly been small in comparison to the characteristic wavelength of the surface acoustic waves, the film constitutes only a small fraction of the volume, where the strain is high. This is no longer the case when the film is not tightly bound to the substrate [26]. When allowing for slippage at the interface between film and substrate, the weakly dispersive quasilongitudinal mode of the film is weakly coupled to the substrate. Its strain field is mainly localized in the film, and the Benjamin-Ono equation was found to be the evolution equation for weakly nonlinear waves of this kind [26]. Porubov and Samsonov have also investigated the general case of a nonlinear film covering a nonlinear substrate [26]. They have derived an evolution equation which they have subsequently reduced to the nonlinear Schrödinger equation to derive approximate stationary periodic solutions.

In the following, we focus on a situation where the film is tightly bound to the substrate, but where the effects of the nonlinearity of the film are still much larger than those of the substrate nonlinearity such that the latter may be neglected. This situation may occur when certain nonlinear elastic moduli of the film have much larger values than those of the substrate. The motivation for this study is twofold. On the one hand, this system leads to interesting new nonlinear evolution equations that deserve to be studied from a mathematical point of view. On the other hand, this system, consisting of a nonlinear film on a linear substrate, is in some respects simpler than the systems with the nonlinearity of the substrate being the dominant one. Because of the linearity of the substrate, the displacement field in the substrate may be formally eliminated. In this way, results may be tested that are obtained by applying standard asymptotic methods to the full equations of motion and corresponding boundary conditions for weakly dispersive surface waves. Solitary surface acoustic waves are essentially two-dimensional objects since their associated displacement field depends on a coordinate parallel to the surface as well as on a depth coordinate. Once the

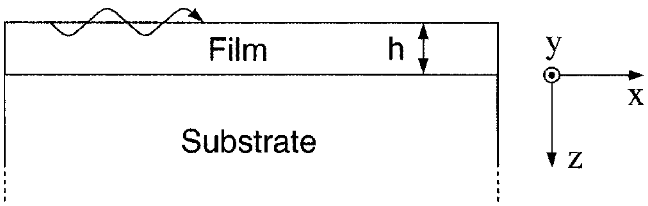

FIG. 1. Geometry.

displacement field is known at the surface, it can be computed at any point in the linear substrate simply by using Green's functions. The analogous problem in the case of a nonlinear substrate would be much more complicated [27,28].

The paper is organized in the following way. In the following section we briefly review the theory of linear surface acoustic waves propagating in an isotropic substrate coated by a thin film. The presence of the film is accounted for by effective boundary conditions [29,30]. In Sec. II, the derivation of these effective boundary conditions is extended to include the second-order nonlinearity of the film and higher-order linear dispersion. With the help of these nonlinear boundary conditions at the surface of the substrate and a traveling-wave ansatz for the sagittal components of the displacement field, the system of equations of motion and boundary conditions for the displacement field is reduced to a single integrodifferential equation for one scalar variable in Sec. III. In Sec. IV, an evolution equation for Rayleigh waves in the system under consideration is derived using a projection method in very much the same way as for Rayleigh waves in nonlinear substrates [5,31]. This procedure is generalized to account for anisotropy of both the film and the substrate material. Using this method, it is then shown that the same evolution equation holds for systems with continuously varying elastic moduli with certain elastic moduli being very large near the surface. With a traveling wave ansatz, this evolution equation is reduced to the integrodifferential equation derived in Sec. III in a more direct approach.

At the end of Sec. IV, the physical system is further generalized to allow for additional nonlinear terms in the effective boundary conditions of Sec. II, that are not generated by the elastic nonlinearity of a covering film. For generalized systems of this kind, a class of evolution equations with nonlocal nonlinearity results. The equation governing propagation of nonlinear Rayleigh waves in a linear substrate covered by a nonlinear film (Sec. IV A) represents a special case in this class.

In Sec. V, the evolution equations derived in the previous sections are analyzed. Numerical solutions are presented for the special case of Sec. IV A corresponding to periodic stationary nonlinear Rayleigh waves. For other special cases, analytic solitary wave and stationary periodic wave solutions are derived, and their depth profiles are discussed. The paper ends with a short conclusion.

I. LINEAR SURFACE WAVES IN LAYERED STRUCTURES

At first let us briefly discuss the linear case. The geometry of the system under consideration is sketched in Fig. 1, where h is the thickness of the coating film, the waves propagate along the x direction, and the z axis is directed normal to the surface into the volume.

The equation of motion for the bulk material in the isotropic case has the form

$$\frac{\partial^2}{\partial t^2}\mathbf{u}=C_t^2\Delta\mathbf{u}+(C_l^2-C_t^2)\text{grad div }\mathbf{u}, \tag{1.1}$$

where $\mathbf{u}$ is the displacement vector, C_t and C_l represent the velocities of transverse and longitudinal bulk waves. The mass density will be called ρ. For simplicity, we shall assume that the material of the coating differs from the substrate only in its density ρ_F. (For the existence of Love waves it is necessary to have a "heavy" covering film with $\rho_F>\rho$.)

Assuming that only sagittal components u_x and u_z of the displacement field are nonzero, one obtains the following effective boundary condition at the surface [29,30]:

$$\rho C_l^2\left(\frac{\partial u_z}{\partial z}+(1-2C_t^2/C_l^2)\frac{\partial u_x}{\partial x}\right)=h(\rho_F-\rho)\frac{\partial^2 u_z}{\partial t^2}, \tag{1.2}$$

$$\rho C_t^2\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right)=h(\rho_F-\rho)\frac{\partial^2 u_x}{\partial t^2},\quad z=0\,. \tag{1.3}$$

In the case of purly shear-horizontal waves we have only one condition, namely, Eq. (1.3), with the displacement component u_x replaced by u_y.

It is convenient to represent the displacement vector $\mathbf{u}$ as the sum $\mathbf{u}=\mathbf{u}^t+\mathbf{u}^l$ of transverse $\mathbf{u}^t$ and longitudinal $\mathbf{u}^l$ parts, which satisfy two-dimensional wave equations. In the case of stationary waves $\mathbf{u}=\mathbf{u}(x-Vt,z)$, propagating in the x direction with the constant velocity V, the functions $\mathbf{u}^t$ and $\mathbf{u}^l$ satisfy the two-dimensional Laplace equation. So for all the components u of vectors $\mathbf{u}^t$ and $\mathbf{u}^l$ we have the following connection between their derivatives at the boundary plane $z=0$:

$$\left.\frac{\partial u}{\partial z}\right|_s=\sqrt{1-V^2/C^2}\hat{H}\left.\frac{\partial u}{\partial x}\right|_s, \tag{1.4}$$

where $C=C_l$ for longitudinal components, $C=C_t$ for transverse components, and $\hat{H}$ denotes the Hilbert transform operator. (The main properties of the Hilbert transform needed for our purposes are listed in Appendix A.) Using the evident connection between the components of $\mathbf{u}^l$ and $\mathbf{u}^t$: $\partial u_x^t/\partial x=-\partial u_z^t/\partial z$ and $\partial u_x^l/\partial z=\partial u_z^l/\partial x$, we can express all components of deformation on the surface in terms of two of them, for example, $v=\partial u_x^l/\partial x$ and $w=\partial u_z^t/\partial x$ (see Appendix B). After substitution of these relations into formulas (1.2),(1.3) we can rewrite them in the simple form,

$$\left[(2-V^2/C_t^2)+d(V/C_t)^2\kappa_t\hat{H}\frac{\partial}{\partial x}\right]w$$
$$+\left[2\kappa_l\hat{H}-d(V/C_t)^2\frac{\partial}{\partial x}\right]v=0,$$

$$\left[(2-V^2/C_t^2)+d(V/C_t)^2\kappa_l\hat{H}\frac{\partial}{\partial x}\right]v-\left[2\kappa_t\hat{H}-d(V/C_t)^2\frac{\partial}{\partial x}\right]w=0, \quad (1.5)$$

where $d=h(\rho_F-\rho)/\rho$, $\kappa_t=\sqrt{1-V^2/C_t^2}$, and $\kappa_l=\sqrt{1-V^2/C_l^2}$. In our long-wave approximation $d\partial/\partial x\sim dk$ is small and in the leading order in this small parameter, the connection between the functions v and w readily follows from Eq. (1.5),

$$w=-\frac{1}{(2-V^2/C_t^2)}\left(2\kappa_l\hat{H}-\frac{1}{2}d\frac{V^4}{C_t^4}\frac{\partial}{\partial x}\right)v, \quad (1.6)$$

and the equation for the dispersion relation,

$$\left[[(2-V^2/C_t^2)^2-4\kappa_l\kappa_t]-d(V/C_t)^4(\kappa_l+\kappa_t)\hat{H}\frac{\partial}{\partial x}\right]v=0, \quad (1.7)$$

which coincides with the corresponding result in Ref. [29] to first order in d.

The first term in Eq. (1.7) gives the velocity C_r of dispersionless Rayleigh waves in a half space without the covering film. The presence of this film gives rise to an effective dispersion of Rayleigh waves, and it follows from Eq. (1.7) that

$$\omega\simeq C_R(k-dk|k|\beta), \quad (1.8)$$

where

$$\beta=\frac{C_R^2}{4C_t^4}\frac{(1-C_R^2/C_t^2)^{-1/2}+(1-C_R^2/C_l^2)^{-1/2}}{(C_t^2-C_R^2)^{-1}+(C_l^2-C_R^2)^{-1}-4(C_t^2-C_R^2)^{-1}}. \quad (1.9)$$

Comparison of this expression with the corresponding expression for shear waves [30],

$$\omega\simeq C_t\left(k-\frac{1}{2}d^2k^3\right), \quad (1.10)$$

shows that the dispersion of Rayleigh waves in the long-wavelength limit is larger than the dispersion of Love waves.

The existence of solitons and their structure in nonlinear evolution systems depends on the competition of nonlinearity and dispersion, and the properties of one-parametric dynamical solitons with a stationary profile strongly depend on the value of the dispersion $D=\partial^2\omega/\partial k^2$ of linear waves in the limit $k\to 0$ [32]. In the case of Rayleigh waves the dispersion in this limit is nonzero: $D=-2C_r\beta d$ in contrast to Love waves and waves in many nonlinear evolution equations of acoustical type (KdV, MKdV, Boussinesq equation, and others) with zero dispersion at the point $k=0$. Consequently, solitary surface acoustic waves of Rayleigh type are expected to differ qualitatively from the soliton solutions of the above equations, but may bear similarities with those of the Benjamin-Ono equation that has the same linear dispersion.

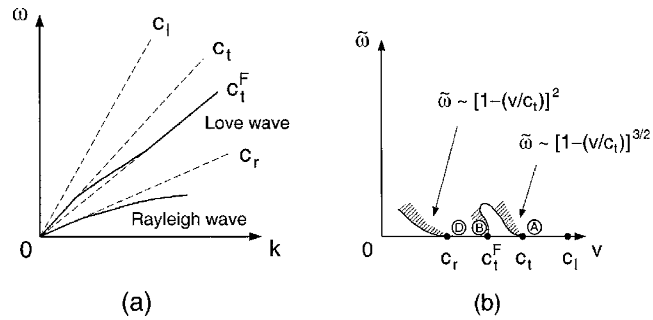

FIG. 2. Linear dispersion relations: $\omega=\omega(k)$ (a), $\tilde{\omega}(V)=\omega(k)-kV$ (b).

The dispersion relations (1.8) and (1.10) are sketched in Fig. 2(a) as a function $\omega=\omega(k)$. Since the frequency ω and wave number k are not convenient characteristics for a solitary wave, let us rewrite the dispersion law in terms of group velocity $V=\partial\omega/\partial k$ and the frequency in the frame of reference moving with this velocity $\tilde{\omega}(k)=\omega(k)-kV$. The dispersion laws (1.8) and (1.10) are shown in Fig. 2(b) as the dependence $\tilde{\omega}=\tilde{\omega}(V)$.

The hatched domains in Fig. 2(b) show the areas where dynamical envelope solitons can exist, and the solid line segments correspond to the one-parametrical solitary waves of stationary profile. But the velocities in segment A are larger than C_t and therefore, the corresponding solitary waves must be unstable with respect to Cherenkov radiation of transverse bulk waves. Surface solitons can exist only in segment DB. The B end of this segment corresponds to the area of large wave numbers in Fig. 2(a) and so solitary waves with velocities V smaller, but close to $C_t^{(F)}$ ($C_t^{(F)}$ is the velocity of transverse waves in the material of the film) must be associated with large field gradients and continuum theory may no longer be appropriate. The solitary waves with velocities V larger, but close to C_r in the D end of the segment DB can exist and represent Rayleigh-type surface solitary waves which are the main object of investigation in this paper.

II. NONLINEAR EFFECTIVE BOUNDARY CONDITIONS AT THE SUBSTRATE SURFACE

If the thickness of the film is much smaller than the characteristic wavelengths of the nonlinear surface waves under consideration, one may expect that the displacement field of the nonlinear wave varies little over the distance between the interface at $z=0$ and the surface at $z=-h$. In this situation, one may apply the method introduced in Refs. [29,30] to eliminate the displacement field inside the thin film in favor of an effective boundary condition at the surface for the displacement field in the substrate. Here, we generalize this method to account for the elastic nonlinearity of the film.

In this section, quantities refering to the film bear a superscript (F) or subscript F while those referring to the substrate are marked by a superscript (S) or subscript S. Expanding the stress tensor at the surface in powers of the film thickness, we may write

$$0=T_{\alpha z}^{(F)}(x,0)-hT_{\alpha z,z}^{(F)}(x,0)+O(h^2) \quad (2.1)$$

and consequently

$$T^{(S)}_{\alpha z}(x,0)=hT^{(F)}_{\alpha z,z}(x,0)+O(h^2), \tag{2.2}$$

where $\alpha=x,z$. In addition, we shall use the equation of motion for the displacement field in the film,

$$T^{(F)}_{\alpha z,z}(x,z)=\rho_F\ddot{u}^{(F)}_{\alpha}-T^{(F)}_{\alpha x,x}(x,z). \tag{2.3}$$

While the displacement field and its derivatives with respect to x and t are continuous at the interface, this is usually not the case for its z derivatives. Therefore, we may replace the time and x derivatives of $u^{(F)}$ on the right-hand side of Eq. (2.2) by the corresponding derivatives of $u^{(S)}$, but not the z derivatives. The latter occur in Eq. (2.2) only as first derivatives and may be expressed in terms of derivatives with respect to x in the following way using the boundary condition at the interface in the form

$$\begin{aligned}C^{(F)}_{\alpha z\mu z}u^{(F)}_{\mu,z}(x,0)=T^{(S)}_{\alpha z}(x,0)-C^{(F)}_{\alpha z\mu x}u^{(F)}_{\mu,x}(x,0)\\ -\frac{1}{2}S^{(F)}_{\alpha z\mu x\nu x}u^{(F)}_{\mu,x}(x,0)u^{(F)}_{\nu,x}(x,0)\\ -S^{(F)}_{\alpha z\mu z\nu x}u^{(F)}_{\mu,z}(x,0)u^{(F)}_{\nu,x}(x,0)\\ -\frac{1}{2}S^{(F)}_{\alpha z\mu z\nu z}u^{(F)}_{\mu,z}(x,0)u^{(F)}_{\nu,z}(x,0).\end{aligned} \tag{2.4}$$

Here, $C^{(F)}_{\alpha\beta\mu\nu}$ are the second-order elastic moduli of the film, while $S^{(F)}_{\alpha\beta\mu\nu\zeta\xi}$ are the following linear combinations of second-order and third-order elastic moduli [33]:

$$S^{(F)}_{\alpha\beta\mu\nu\zeta\xi}=C^{(F)}_{\alpha\beta\mu\nu\zeta\xi}+\delta_{\alpha\mu}C^{(F)}_{\beta\nu\zeta\xi}+\delta_{\alpha\zeta}C^{(F)}_{\beta\xi\mu\nu}+\delta_{\mu\zeta}C^{(F)}_{\alpha\beta\nu\xi}. \tag{2.5}$$

The first term on the right-hand side of Eq. (2.4) can be neglected as it is of first order in h. Equation (2.4) may be solved for the quantities $u^{(F)}_{\mu,z}(x,0)$ by iteration to second order in the x derivatives of the displacement field. These solutions may be inserted into the right-hand side of Eq. (2.2), and the $u^{(F)}_{\mu,x}(x,0)$ may be replaced by $u^{(S)}_{\mu,x}(x,0)$. Confining ourselves to terms of first order in h and second order in the displacement gradients, we obtain the following effective boundary condition:

$$\begin{aligned}T^{(S)}_{\alpha z}(x,0)=h\{\rho_F\ddot{u}^{(S)}_{\alpha}(x,0)-\bar{S}^{(F)}_{\alpha x\mu x}u^{(S)}_{\mu,xx}(x,0)\\ -\bar{S}^{(F)}_{\alpha x\mu x\nu x}u^{(S)}_{\mu,x}(x,0)u^{(S)}_{\nu,xx}(x,0)\}+O(h^2)+O(u^3).\end{aligned} \tag{2.6}$$

The definitions of the coupling coefficients occurring on the right-hand side of Eq. (2.6) are given below,

$$\bar{S}^{(F)}_{\alpha x\mu x}=C^{(F)}_{\alpha x\mu x}-C^{(F)}_{\alpha x\beta z}\Gamma_{\beta\gamma}C^{(F)}_{\gamma z\mu x}, \tag{2.7}$$

$$\begin{aligned}\bar{S}^{(F)}_{\alpha x\mu x\nu x}=S^{(F)}_{\alpha x\mu x\nu x}-\Sigma^{(1)}_{\alpha\mu\nu}-\Sigma^{(1)}_{\mu\alpha\nu}-\Sigma^{(1)}_{\nu\mu\alpha}+\Sigma^{(2)}_{\alpha\mu\nu}+\Sigma^{(2)}_{\mu\alpha\nu}\\ +\Sigma^{(2)}_{\nu\mu\alpha}-S^{(F)}_{\beta z\lambda z\zeta z}\Gamma_{\beta\gamma}C^{(F)}_{\gamma z\alpha x}\Gamma_{\lambda\sigma}C^{(F)}_{\sigma z\mu x}\Gamma_{\zeta\xi}C^{(F)}_{\xi z\nu x},\end{aligned} \tag{2.8}$$

where

$$\Sigma^{(1)}_{\alpha\mu\nu}=S^{(F)}_{\alpha x\mu x\zeta z}\Gamma_{\zeta\xi}C^{(F)}_{\xi z\nu x}, \tag{2.9}$$

$$\Sigma^{(2)}_{\alpha\mu\nu}=S^{(F)}_{\alpha x\beta z\zeta z}\Gamma_{\beta\gamma}C^{(F)}_{\gamma z\mu x}\Gamma_{\zeta\xi}C^{(F)}_{\xi z\nu x} \tag{2.10}$$

and $(\Gamma_{\alpha\beta})$ is the inverse of the 3×3 matrix $(C^{(F)}_{\alpha z\beta z})$.

The derivation of effective boundary conditions outlined above may be carried to higher orders in the film thickness h in a straightforward way. The linear term of second order in h on the right-hand side of Eq. (2.6) is given by

$$\begin{aligned}-\frac{1}{2}h^2\{[C^{(F)}_{\alpha x\mu z}\Gamma_{\mu\nu}-\Gamma_{\alpha\beta}C^{(F)}_{\beta z\nu x}]\rho_F\ddot{u}_{\nu,x}+[C^{(F)}_{\alpha x\mu x}\Gamma_{\mu\nu}C^{(F)}_{\nu z\gamma x}\\ +C^{(F)}_{\alpha x\mu z}\Gamma_{\mu\nu}C^{(F)}_{\nu x\gamma x}-C^{(F)}_{\alpha x\mu z}\Gamma_{\mu\beta}(C^{(F)}_{\beta z\nu x}+C^{(F)}_{\beta x\nu z})\Gamma_{\nu\delta}C^{(F)}_{\delta z\gamma x}\\ -2C^{(F)}_{\alpha x\mu z}\Gamma_{\mu\nu}\bar{S}^{(F)}_{\nu x\gamma x}]u_{\gamma,xxx}\}.\end{aligned} \tag{2.11}$$

In the following, we shall suppress the superscript (S) at the displacement field. The boundary conditions (2.6) may be applied to any anisotropic nonpiezoelectric media. If the film material is isotropic, they greatly simplify and become

$$\begin{aligned}T^{(S)}_{xz}(x,0)=h\Bigg(\rho_F\ddot{u}_x(x,0)-\left[\lambda_F+2\mu_F-\frac{\lambda_F^2}{\lambda_F+2\mu_F}\right]u_{x,xx}(x,0)\\ -\frac{1}{2}\Bigg\{(2C_F+6B_F+2A_F+3\lambda_F+6\mu_F)\\ \times\left[1-\left(\frac{\lambda_F}{\lambda_F+2\mu_F}\right)^3\right]-3(2C_F+2B_F+\lambda_F)\\ \times\frac{\lambda_F}{\lambda_F+2\mu_F}\left(1-\frac{\lambda_F}{\lambda_F+2\mu_F}\right)\Bigg\}\frac{\partial}{\partial x}u^2_{x,x}\\ -\frac{1}{2}\left[2(\lambda_F+\mu_F)\left(1-\frac{\lambda_F}{\lambda_F+2\mu_F}\right)\right]\frac{\partial}{\partial x}u^2_{z,x}\Bigg)\\ -\frac{1}{2}h^2\Bigg\{\frac{-2\mu_F}{\lambda_F+2\mu_F}\rho_F\ddot{u}_{z,x}+\Bigg[\lambda_F+2\mu_F\\ -\frac{\lambda_F^2}{\lambda_F+2\mu_F}\Bigg]u_{z,xxx}\Bigg\},\end{aligned} \tag{2.12}$$

$$\begin{aligned}T^{(S)}_{zz}(x,0)=h\Bigg\{\rho_F\ddot{u}_z(x,0)-2(\lambda_F+\mu_F)\\ \times\left(1-\frac{\lambda_F}{\lambda_F+2\mu_F}\right)\frac{\partial}{\partial x}[u_{z,x}(x,0)u_{x,x}(x,0)]\Bigg\}\\ +\frac{1}{2}h^2\Bigg\{\frac{-2\mu_F}{\lambda_F+2\mu_F}\rho_F\ddot{u}_{x,x}+\Bigg[\lambda_F+2\mu_F\\ -\frac{\lambda_F^2}{\lambda_F+2\mu_F}\Bigg]u_{x,xxx}\Bigg\}.\end{aligned} \tag{2.13}$$

Here, λ and μ are the (linear) Lamé constants, and A,B,C are third-order elastic moduli of an isotropic elastic medium defined in Ref. [35].

When applying these boundary conditions, one has to keep in mind that the nonlinear terms on the right-hand side of Eq. (2.6) are more important than the corresponding nonlinear terms in $(T^{(S)}_{\alpha\beta})$ if at least one of the nonlinear elastic moduli A_F, B_F, or C_F is much larger than the corresponding value of the substrate material. This means that one normally has to keep only the first nonlinear term on the right-hand side of Eq. (2.12) and may neglect the nonlinear term in Eq. (2.13).

III. NONLINEAR TRAVELING WAVE ANSATZ

In Eq. (1.2) we now replace the right-hand side by the right-hand sides of Eqs. (2.12) and (2.13). When searching for nonlinear wave solutions with stationary profile of the form $\mathbf{u}=\mathbf{u}(x-Vt,z)$ we can use the connection between the components of deformation at the surface (B1) and rewrite the two boundary conditions in the form

$$\left\{2-(V/C_t)^2-h\frac{\rho_F V^2-\gamma_0}{\rho C_t^2}\kappa_t\hat{H}\frac{\partial}{\partial x}\right\}w+\left\{2\kappa_l\hat{H}+h\frac{\rho_F V^2-\gamma_0}{\rho C_t^2}\frac{\partial}{\partial x}\right\}v=\frac{1}{2}h\left\{\frac{\gamma_1}{\rho C_t^2}\frac{\partial}{\partial x}[v-\kappa_t\hat{H}w]^2+\frac{\gamma_2}{\rho C_t^2}\frac{\partial}{\partial x}[\kappa_l\hat{H}v+w]^2\right\}, \tag{3.1}$$

$$\left\{2\kappa_t\left(\frac{C_t}{C_l}\right)^2\hat{H}+h\frac{\rho_F V^2}{\rho C_l^2}\frac{\partial}{\partial x}\right\}w+\left\{\left(\frac{V}{C_l}\right)^2-2\left(\frac{C_t}{C_l}\right)^2+h\frac{\rho_F V^2}{\rho C_l^2}\kappa_l\hat{H}\frac{\partial}{\partial x}\right\}v=h\frac{\gamma_2}{\rho C_l^2}\frac{\partial}{\partial x}[\kappa_l\hat{H}v+w][v-\kappa_t\hat{H}w], \tag{3.2}$$

where we have introduced the coefficients

$$\gamma_0=\lambda+2\mu-\frac{\lambda^2}{\lambda+2\mu}, \tag{3.3}$$

$$\gamma_1=(2A_F+6B_F+2C_F+3\lambda+6\mu)\left[1-\left(\frac{\lambda}{\lambda+2\mu}\right)^3\right]-3(2B_F+2C_F+\lambda)\frac{\lambda}{\lambda+2\mu}\left[1-\frac{\lambda}{\lambda+2\mu}\right], \tag{3.4}$$

$$\gamma_2=2(\lambda+\mu)\left[1-\frac{\lambda}{\lambda+2\mu}\right]. \tag{3.5}$$

For simplicity, we assume here $\lambda_F=\lambda_S$, $\mu_F=\mu_S$ and suppress the indices F and S at the linear Lamé constants. The first of the two equations (3.1) and (3.2) can be solved for w up to first order in the film thickness h,

$$w=-\frac{2\kappa_l}{2-(V/C_t)^2}\hat{H}v-\frac{h}{2-(V/C_t)^2}\left\{\frac{\rho_F V^2-\gamma_0}{\rho C_t^2}\left[1-\frac{2\kappa_l\kappa_t}{2-(V/C_t)^2}\right]\frac{\partial}{\partial x}v-\frac{\gamma_1}{2\rho C_t^2}\left[1-\frac{2\kappa_l\kappa_t}{2-(V/C_t)^2}\right]^2\frac{\partial}{\partial x}v^2-\frac{\gamma_2}{2\rho C_t^2}\times\kappa_l^2\left[1-\frac{2}{2-(V/C_t)^2}\right]^2\frac{\partial}{\partial x}(\hat{H}v)^2\right\}+O(h^2). \tag{3.6}$$

Inserting this into Eq. (3.2) leads to the following integro-differential equation for the variable v:

$$\left\{D(V)-h(V/C_t)^2\frac{\rho_F-\rho}{\rho}(\kappa_l+\kappa_t)\hat{H}\frac{\partial}{\partial x}\right\}v=-h\frac{\kappa_t}{[2-(V/C_t)^2]^2}\left\{\frac{\gamma_1}{\mu}[2-(V/C_t)^2-2\kappa_l\kappa_t]^2+\frac{\gamma_2}{\mu}\kappa_l\kappa_t(V/C_t)^4\right\}\hat{H}\frac{\partial}{\partial x}v^2. \tag{3.7}$$

Here, $D(V)=4\kappa_l\kappa_t-[(V/C_t)^2-2]^2$ is the Rayleigh determinant, which is a small quantity, as the relative deviation $\varepsilon=(V-C_r)/C_t$ from the velocity of linear Rayleigh waves C_r is small,

$$D(V)\simeq-\nu\varepsilon\equiv-\varepsilon 4\xi\left[\frac{\kappa_l^0}{\kappa_t^0}+\left(\frac{C_t}{C_l}\right)^2\frac{\kappa_t^0}{\kappa_l^0}+\xi^2-2\right]>0, \tag{3.8}$$

where $\xi=C_R/C_t$, $\kappa_t^0=\sqrt{1-\xi^2}$, $\kappa_l^0=\sqrt{1-\xi^2C_t^2/C_l^2}2$. In the derivation of Eq. (3.7) we have confined ourselves to terms up to first order in the film thickness h in the effective boundary conditions (2.12),(2.13). If the h^2 terms linear in the displacement field are taken into account, the term

$$h^2K\frac{\partial^2}{\partial x^2}v \tag{3.9}$$

has to be added to the right-hand side of Eq. (3.7). For simplicity, we have assumed in the derivation of Eq. (3.7) that the linear Lamé constants of film and substrate are identical. If in addition to the mass densities, the Lamé constants of the film differ from those of the substrate, an intregrodifferential equation for v will be obtained that has the same form as Eq. (3.7) with partly different coefficients. In this more general case, the coefficient in front of the term linear in v and of first order in h may vanish. More precisely, this happens when [34]

$$\rho_F/\rho_S=c_F/c_S, \tag{3.10}$$

where $c=\lambda+2\mu-\lambda^2/(\lambda+2\mu)$. In this case, the coefficient K has the form

$$K=\frac{1}{2}\xi^2(2-\xi^2)\frac{c_F}{c_S}\left[\frac{c_S}{\mu_S}-\frac{2\mu_F}{\lambda_F+2\mu_F}\right.$$
$$\left.-[4-2(C_t/C_l)^2(\xi^2+2)]\frac{c_F}{c_S}\right], \qquad (3.11)$$

where again $\xi=C_r/C_t$ and C_t, C_l, and C_r are the velocities of bulk shear, bulk longitudinal, and Rayleigh waves of the substrate. Depending on the Lamé constants of the film, K can be positive or negative.

After rescaling of the coordinate x and field variable v, Eq. (3.7) assumes the form

$$\tilde{U}-\beta_1\hat{H}\tilde{U}_\eta-\beta_2\tilde{U}_{\eta\eta}-\alpha_3\hat{H}(\tilde{U}\tilde{U}_\eta)=0, \qquad (3.12)$$

with dimensionless coefficients β_1, β_2, and α_3, and $\eta\propto x-Vt$ and $\tilde{U}$ being dimensionless, too. Already at this stage, we may note that Eq. (3.12) has the unusual feature of a nonlocal linear dispersion term *and* a nonlocal nonlinearity. The lowest-order linear dispersion term is obviously that of the Benjamin-Ono equation, which reduces to

$$\tilde{U}-\hat{H}\tilde{U}_\eta-\tilde{U}^2=0, \qquad (3.13)$$

with a traveling wave ansatz. This may be compared with the KdV case,

$$\tilde{U}-\tilde{U}_{\eta\eta}-\tilde{U}^2=0, \qquad (3.14)$$

where both the dispersion term and the nonlinearity are local.

IV. DERIVATION OF AN EVOLUTION EQUATION

A. Boundary conditions corresponding to a nonlinear film

A convenient way of deriving evolution equations for nonlinear guided acoustic waves is the projection method [31]. Anisotropy may be accounted for with virtually no additional complication, and the application of the method to a linear substrate with nonlinear effective boundary conditions at the surface is particularly easy.

We start with an asymptotic expansion of the diplacement field

$$\mathbf{u}(x,z,t)=\varepsilon\mathbf{u}^{(1)}(x,z,t)+\varepsilon^{3/2}\mathbf{u}^{(2)}(x,z,t)+O(\varepsilon^2), \qquad (4.1)$$

with a dimensionless expansion parameter $\varepsilon\ll 1$. We also invoke the scaling $h=O(\varepsilon^{1/2})$ for the film thickness and $\overline{S^{(F)}_{\alpha x\mu x\nu x}}=O(1/\varepsilon)$ for (at least some of) the nonlinear coefficients of the film material. This particular scaling has been chosen to make sure that the effects of the film are larger (and hence appear at lower order of ε) than the nonlinearity of the substrate.

The first-order field $\mathbf{u}^{(1)}$ has to satisfy the equations of motion

$$\rho_S\ddot{u}^{(1)}_\alpha(x,z,t)=C^{(S)}_{\alpha\beta\mu\nu}u^{(1)}_{\mu,\beta\nu}(x,z,t) \qquad (4.2)$$

subject to the boundary conditions

$$C^{(S)}_{\alpha z\mu\nu}u^{(1)}_{\mu,\nu}(x,0,t)=0. \qquad (4.3)$$

A solution of these equations may be constructed as a superposition of Rayleigh waves with amplitudes that are allowed to depend on a "slow" time variable $\tau=\varepsilon^{1/2}t$:

$$\mathbf{u}^{(1)}=\sum_q e^{iq(x-C_rt)}\mathbf{w}(z|q)A_q(\tau). \qquad (4.4)$$

The sum over q in Eq. (4.4) runs over integer multiples of $2\pi/L$, where L is an arbitrary spatial periodicity. $\mathbf{w}(z|q)$ is the depth profile of a linear Rayleigh wave with wave number q. It is a superposition of up to three exponentials in nonpiezoelectric media. In the case of an isotropic substrate and $q>0$, it has the familiar form

$$\mathbf{w}(z|q)=\begin{pmatrix}1\\0\\i\kappa_l^{(0)}\end{pmatrix}\exp(-q\kappa_l^{(0)}z)$$
$$+K_t\begin{pmatrix}1\\0\\i/\kappa_t^{(0)}\end{pmatrix}\exp(-q\kappa_t^{(0)}z), \qquad (4.5)$$

with the coefficient

$$K_t=-\sqrt{\kappa_l^{(0)}\kappa_t^{(0)}}. \qquad (4.6)$$

At order $O(\varepsilon^{3/2})$, we obtain from the equation of motion in the substrate,

$$\rho_S\ddot{u}^{(2)}_\alpha(x,z,t)-C^{(S)}_{\alpha\beta\mu\nu}u^{(2)}_{\mu,\beta\nu}(x,z,t)$$
$$=\sum_q 2iC_rq\rho_Sw_\alpha(z|q)\frac{\partial}{\partial\tau}A_q(\tau)e^{iq(x-C_rt)}, \qquad (4.7)$$

and from the boundary conditions (2.6)

$$C^{(S)}_{\alpha z\mu\nu}u^{(2)}_{\mu,\nu}(x,0,t)$$
$$=h\{\rho_FC_r^2\delta_{\alpha\mu}-\overline{S^{(F)}_{\alpha x\mu x}}\}\sum_q q^2w_\mu(0|q)A_q(\tau)e^{iq(x-C_rt)}$$
$$-(h/2)\overline{S^{(F)}_{\alpha x\mu x\nu x}}\sum_{q,q'}iqq'(q+q')$$
$$\times w_\mu(0|q)w_\mu(0|q')A_q(\tau)A_{q'}(\tau). \qquad (4.8)$$

We now multiply Eq. (4.7) by $w^*_\alpha(z|k)\exp[-ik(x-C_rt)]$, sum over α, integrate over the area $0<x<L$, $0<z<\infty$, apply Green's second integral theorem and make use of the boundary conditions (4.3),(4.8). With the transformation $A_q(\tau)=B_q(\tau)/(iq)$, we finally obtain

$$iN\frac{\partial}{\partial\tau}B_k=-k^2\tilde{\beta}_1B_k+k^2\sum_q\overline{\alpha_3}(-k,q,k-q)B_qB_{k-q}. \qquad (4.9)$$

The coefficients in Eq. (4.9) have the explicit form

$$N=2C_r\rho_S k\int_0^\infty |\mathbf{w}(z|k)|^2\,dz, \tag{4.10}$$

which is independent of k for an appropriate normalization of **w**;

$$\tilde{\beta}_1=h[\rho_F C_r^2 w_\alpha^*(0|k)w_\alpha(0|k)-\overline{S}^{(F)}_{\alpha x\mu x}w_\alpha^*(0|k)w_\mu(0|k)] \tag{4.11}$$

is also independent of k. The quantities

$$\overline{\alpha}_3(-k,q,k-q)=h\overline{S}^{(F)}_{\lambda x\mu x\nu x}w_\lambda^*(0|k)w_\mu(0|q)w_\nu(0|k-q) \tag{4.12}$$

depend only on the signs of the wave numbers k, q, and $k-q$, and they may be complex. Defining the coefficient $\tilde{\alpha}_3=\overline{\alpha}_3(-k,q,k-q)$ for $k>0$, $0<q<k$, we may write for $k>0$,

$$\sum_q \overline{\alpha}_3(-k,q,k-q)B_qB_{k-q}=\tilde{\alpha}_3\sum_{0<q<k}B_qB_{k-q}+2\tilde{\alpha}_3^*\sum_{q>k}B_qB^*_{q-k}, \tag{4.13}$$

where we have made use of the reality of the displacement field via $B_{-k}=B_k^*$. It is then evident that the coefficient $\tilde{\alpha}_3$ can be made real by the simple transformation $B_k\to B_k\exp(i\phi)$ with an appropriate phase angle ϕ. Defining now the scalar field $U(\xi,\tau)$ via

$$U(\xi,\tau)=\sum_q B_q(\tau)e^{iq\xi}, \tag{4.14}$$

where $\xi=x-C_rt$, the evolution equation (4.9) can be brought into the real-space form

$$U_\tau+\beta_1\hat{H}U_{\xi\xi}+\alpha_3\hat{H}(UU_\xi)_\xi=0. \tag{4.15}$$

We emphasize that this evolution equation applies to substrate and film materials of arbitrary anisotropy. Making now a traveling wave ansatz $U(\xi,\tau)=\tilde{U}(\xi-v\tau)$ with the velocity shift $V-C_r=\varepsilon^{1/2}v$, we are led to the integrodifferential equation (3.12), when $\eta=\xi-v\tau$.

If the coefficient β_1 vanishes, a modification of the derivation has to be applied to include higher-order dispersion in the evolution equation. We now apply the scaling $h=O(\varepsilon^{1/4})$ and $\overline{S}^{(F)}_{\alpha x\mu x\nu x}=O(1/\varepsilon^{3/4})$. The displacement field is expanded as

$$\mathbf{u}(x,z,t)=\varepsilon\mathbf{u}^{(1)}(x,z,t)+\varepsilon^{5/4}\mathbf{u}^{(2)}(x,z,t)+\varepsilon^{3/2}\mathbf{u}^{(3)}(x,z,t)+O(\varepsilon^2). \tag{4.16}$$

It is an easy task to determine the field $\mathbf{u}^{(2)}$ explicitly. The boundary conditions at order $O(\varepsilon^{5/4})$ merely lead to a modification of the coefficients in front of the exponentials that constitute $\mathbf{w}(z|q)$. In the isotropic case, we may write

$$\mathbf{u}^{(3/2)}(x,z,t)=\Delta K_t\sum_q e^{iq(x-C_rt)}q\begin{pmatrix}1\\0\\i/\kappa_t^{(0)}\end{pmatrix}\times\exp(-q\kappa_t^{(0)}z)A_q(\tau), \tag{4.17}$$

with the coefficient

$$\Delta K_t=\frac{h[\rho_FC_r^2-c_F]}{2\mu}\,\frac{\kappa_t^{(0)}[1-(\kappa_t^{(0)})^2]}{1+(\kappa_t^{(0)})^2}. \tag{4.18}$$

At order $O(\varepsilon^{3/2})$ the terms proportional to h^2 of the effective boundary conditions come into play as well as the field $\mathbf{u}^{(2)}$ in connection with the terms proportional to h in the boundary conditions. One may proceed in the same way as in the above application of the projection method to arrive at the evolution equation

$$U_\tau+\beta_2U_{\xi\xi\xi}+\alpha_3\hat{H}(UU_\xi)_\xi=0, \tag{4.19}$$

which has the same linear dispersion as the KdV equation, but it still has a nonlocal nonlinearity.

B. Continuously varying material properties

In the derivation of the evolution equations (4.15) and (4.19) we have so far considered a thin nonlinear film which has a sharp interface with a homogeneous substrate. The presence of the film has been accounted for by the effective boundary condition (2.6),(2.11). We now show that these equations also govern nonlinear surface wave propagation in a medium with continuously varying material properties near the surface and a large second-order nonlinearity in the neighborhood of the surface.

The system is described by the Lagrangian

$$L=\int d^3x\left\{\frac{1}{2}\rho\dot{u}_\alpha\dot{u}_\alpha-\frac{1}{2}S_{\alpha\beta\mu\nu}u_{\alpha,\beta}u_{\mu,\nu}-\frac{1}{6}S_{\alpha\beta\mu\nu\zeta\xi}u_{\alpha,\beta}u_{\mu,\nu}u_{\zeta,\xi}\right\}. \tag{4.20}$$

The mass density ρ and coupling coefficients $S_{\alpha\beta\mu\nu}$ and $S_{\alpha\beta\mu\nu\zeta\xi}$ are now allowed to be functions of z. These functions are assumed to be differentiable for $z>0$. At $z=0$, they have a discontinuity and they vanish for $z<0$.

From Hamilton's principle, the following equations of motion in the medium and boundary conditions at the surface are derived:

$$\rho(z)\ddot{u}_\alpha(x,z)=\frac{\partial}{\partial x_\beta}S_{\alpha\beta\mu\nu}(z)u_{\mu,\nu}(x,z)+\frac{1}{2}\frac{\partial}{\partial x_\beta}S_{\alpha\beta\mu\nu\zeta\xi}(z)u_{\mu,\nu}(x,z)u_{\zeta,\xi}(x,z)=0, \tag{4.21}$$

$$S_{\alpha z\mu\nu}(0)u_{\mu,\nu}(x,0)+\frac{1}{2}S_{\alpha z\mu\nu\zeta\xi}(0)u_{\mu,\nu}(x,0)u_{\zeta,\xi}(x,0)=0. \tag{4.22}$$

Introducing again a dimensionless expansion parameter $\varepsilon \ll 1$, we assume that the coefficients in the above equations may be decomposed in the following way:

$$\rho(z)=\rho_0+\varepsilon^{1/2}\rho_1(z), \tag{4.23}$$

$$S_{\alpha\beta\mu\nu}(z)=C_{\alpha\beta\mu\nu}+\varepsilon^{1/2}\delta S_{\alpha\beta\mu\nu}(z), \tag{4.24}$$

$$S_{\alpha\beta\mu\nu\zeta\xi}(z)=\overline{S}_{\alpha\beta\mu\nu\zeta\xi}+\varepsilon^{-1/2}\delta S_{\alpha\beta\mu\nu\zeta\xi}(z). \tag{4.25}$$

With this choice of scaling, the linear dispersion and the nonlinear terms associated with $\delta S_{\alpha\beta\mu\nu\zeta\xi}(z)$ will be of the same order of ε, while the "background" nonlinearity with coefficients $\overline{S}_{\alpha\beta\mu\nu\zeta\xi}$ will appear at higher orders of ε.

In an asymptotic expansion (4.1) of the displacement field, the first-order field $\mathbf{u}^{(1)}$ has to satisfy the equations of motion

$$\rho_0\ddot{u}_\alpha^{(1)}(x,z,t)=C_{\alpha\beta\mu\nu}u_{\mu,\beta\nu}^{(1)}(x,z,t) \tag{4.26}$$

subject to the boundary conditions

$$C_{\alpha z\mu\nu}u_{\mu,\nu}^{(1)}(x,0,t)=0. \tag{4.27}$$

A solution of these equations may again be constructed as a superposition of Rayleigh waves (4.4) with amplitudes that are allowed to depend on a "slow" time variable $\tau=\varepsilon^{1/2}t$.

At order $O(\varepsilon^{3/2})$, we obtain

$$\begin{aligned}\rho_0\ddot{u}_\alpha^{(2)}(x,z,t)-C_{\alpha\beta\mu\nu}u_{\mu,\beta\nu}^{(2)}(x,z,t)\\ =\sum_q\Big\{2iC_rq\rho_0w_\alpha(z|q)\frac{\partial}{\partial\tau}+\rho_1(z)(C_rq)^2w_\alpha(z|q)\\ +D_\beta(q)\,\delta S_{\alpha\beta\mu\nu}(z)D_\nu(q)w_\mu(z|q)\Big\}A_q(\tau)e^{iq(x-C_Rt)}\\ +\sum_{q,k}D_\beta(q+k)\,\delta S_{\alpha\beta\mu\nu\zeta\xi}(z)[D_\nu(q)w_\mu(z|q)]\\ \times[D_\xi(k)w_\zeta(z|k)]A_q(\tau)A_k(\tau)e^{i(q+k)(x-C_Rt)},\end{aligned} \tag{4.28}$$

$$\begin{aligned}C_{\alpha z\mu\nu}u_{\mu,\nu}^{(2)}(x,0,t)\\ =-\delta S_{\alpha z\mu\nu}(0)\sum_q[D_\nu(q)w_\mu(z|q)]_{z=0}A_q(\tau)\\ \times e^{iq(x-C_Rt)}-\sum_{q,k}\delta S_{\alpha\beta\mu\nu\zeta\xi}(0)[D_\nu(q)w_\mu(z|q)]_{z=0}\\ \times[D_\xi(k)w_\zeta(z|k)]_{z=0}A_q(\tau)A_k(\tau)e^{i(q+k)(x-C_Rt)}.\end{aligned} \tag{4.29}$$

In Eqs. (4.28) and (4.29), we have introduced the operator $D_\alpha(q)=\delta_{\alpha x}iq+\delta_{\alpha z}\partial/\partial z$.

We now apply the projection method, i.e., we multiply the right-hand and left-hand sides of Eq. (4.28) by $w_\alpha^*(z|q)\exp[-iq(x-C_Rt)]$, sum over α, integrate over x from 0 to L, and over z from 0 to ∞. When integrating by parts and making use of the boundary conditions (4.29) and (4.27), we are led to the evolution equation

$$N\Big\{i\frac{\partial}{\partial\tau}+\Delta(q)\Big\}A_q=\sum_k K(-q,k,q-k)A_kA_{q-k}, \tag{4.30}$$

where N is given by Eq. (4.10) with ρ_S replaced by ρ_0,

$$\begin{aligned}N\Delta(q)=\int_0^\infty dz\{\rho_1(z)(C_Rq)^2w_\alpha^*(z|q)w_\alpha(z|q)-\delta S_{\alpha\beta\mu\nu}(z)\\ \times[D_\beta(q)w_\alpha(z|q)]^*[D_\nu(q)w_\mu(z|q)]\},\end{aligned} \tag{4.31}$$

$$\begin{aligned}K(q_1,q_2,q_3)=\int_0^\infty dz\,\delta S_{\alpha\beta\mu\nu\zeta\xi}(z)[D_\beta(q_1)w_\alpha(z|q_1)]\\ \times[D_\nu(q_2)w_\mu(z|q_2)][D_\xi(q_3)w_\zeta(z|q_3)].\end{aligned} \tag{4.32}$$

The evolution equation (4.30) with the coefficients defined in Eqs. (4.10), (4.31), and (4.32) is of quite general validity and may be applied to anisotropic media with depth-dependent material parameters. The linear dispersion law is given by the function $\Delta(q)$. Depending on the depth profile of the mass density and second-order elastic moduli, Δ may depend on the wave number q in a complicated way. The linear dispersion term in Eq. (4.30) may be regarded as a solid state analogue of the dispersion term in the intermediate long-wave equation [40] arising in fluid dynamics. When expanding $\Delta(q)$ in powers of q for small q, the leading term is of at least second order.

To establish the connection between Eq. (4.30) derived for continuously varying material parameters and our previous considerations for a nonlinear film covering a linear substrate, we now assume that the z-dependent parts of the material parameters are strongly localized at the surface, for example,

$$\delta S_{\alpha\beta\mu\nu\zeta\xi}(z)=\tilde{S}_{\alpha\beta\mu\nu\zeta\xi}be^{-bz}, \tag{4.33}$$

$$\delta S_{\alpha\beta\mu\nu}(z)=\delta\tilde{S}_{\alpha\beta\mu\nu}be^{-bz}, \tag{4.34}$$

$$\rho_1(z)=\tilde{\rho}_1be^{-bz}, \tag{4.35}$$

with $1/b$ being much smaller than the penetration depth of Rayleigh waves. In this case, the coefficients $N\Delta_q$ and K take the form

$$\begin{aligned}N\Delta(q)=\tilde{\rho}_1(C_Rq)^2w_\alpha^*(0|q)w_\alpha(0|q)\\ -\delta\tilde{S}_{\alpha\beta\mu\nu}[D_\beta(q)w_\alpha(z|q)]_{z=0}^*\\ \times[D_\nu(q)w_\mu(z|q)]_{z=0},\end{aligned} \tag{4.36}$$

$$K(q_1,q_2,q_3)=\widetilde{S}_{\alpha\beta\mu\nu\zeta\xi}[D_\beta(q_1)w_\alpha(z|q_1)]^*_{z=0}\times[D_\nu(q_2)w_\mu(z|q_2)]_{z=0}\times[D_\xi(q_3)w_\zeta(z|q_3)]_{z=0}. \quad (4.37)$$

By using the linear boundary conditions satisfied by **w**, the z derivatives of **w** at $z=0$ may be eliminated in the expressions (4.36) and (4.37) for the coefficients $N\Delta_q$ and $K(q_1,q_2,q_3)$,

$$\left\{\frac{\partial}{\partial z}w_\alpha(z|q)\right\}_{z=0}=-\Gamma_{\alpha\beta}C_{\beta z\mu x}iqw_\mu(0|q). \quad (4.38)$$

Keeping in mind that the depth-dependent contribution $\delta S_{\alpha\beta\mu\nu}(z)$ to the second-order elastic moduli is a correction of order $O(\varepsilon^{1/2})$, one may replace in Eq. (4.38) the elastic moduli C by $C+\delta S$. In this way, the evolution equation (4.15) and the results for the coefficients occuring therein are recovered.

We briefly indicate that the evolution equation (4.15) is also obtained in a continuum description of an adsorbate layer strongly bound to the surface [36] with a strong in-plane nonlinearity. This description leads to boundary conditions for the displacement field in the substrate which have essentially identical form as the ones for a nonlinear film.

C. Generalization of the effective boundary conditions

The derivations of the evolution equations (4.15),(4.19) and likewise the integrodifferential equation (3.12) are based on the autonomous system of an elastic half space with a free surface, having modified elastic properties near the surface, in particular strong second-order nonlinearity. At the end of this section, we further generalize boundary condition (2.6) with the term (2.11). We do this by replacing the nonlinear term $-h\overline{S}^{(F)}_{\alpha x\mu x\nu x}u^{(S)}_{\mu,x}(x,0)u^{(S)}_{\nu,xx}(x,0)$ in Eq. (2.6) by the more general expression $\Lambda_{\alpha\beta\mu\nu\zeta\xi}u^{(S)}_{\mu,\nu}(x,0)u^{(S)}_{\zeta,\xi\beta}(x,0)$. Although this extension may not be of direct physical relevance, it contains interesting aspects from the mathematical point of view as will become clear in the following section. For simplicity, we take the linear substrate to be isotropic. The Cartesian indices then only run over x and z. For the tensor $(\Lambda_{\alpha\beta\mu\nu\zeta\xi})$ we do not require any symmetry property except that the indices x and z occur in even numbers. When applying now the projection method in precisely the same way as in Sec. IV A, but with the new generalized boundary condition instead of Eq. (2.6), we obtain the evolution equation

$$iN\frac{\partial}{\partial\tau}B_k=-k^2\widetilde{\beta}_1B_k-\frac{1}{2}k\left\{\sum_{0<q<k}k\kappa_1B_qB_{k-q}+\sum_{q>k}[k\kappa_3-q(\kappa_2+\kappa_3)]B_qB^*_{q-k}\right\}. \quad (4.39)$$

The coefficients $\kappa_1,\kappa_2,\kappa_3$ can be expressed explicitly in terms of the tensor $(\Lambda_{\alpha\beta\mu\nu\zeta\xi})$ and the displacement field of linear Rayleigh waves,

$$\kappa_1=i\Lambda_{\alpha\beta\mu\nu\zeta\xi}w^*_\alpha(0|k)\times[D_\nu(k)w_\mu(z|k)]_{z=0}[D_\beta(k)D_\xi(k)w_\zeta(z|k)]_{z=0}/k^3,$$
$$\kappa_2=i\Lambda_{\alpha\beta\mu\nu\zeta\xi}w^*_\alpha(0|k)\times[D_\nu(k)w_\mu(z|k)]_{z=0}[D_\beta(k)D_\xi(k)w_\zeta(z|k)]^*_{z=0}/k^3,$$
$$\kappa_3=i\Lambda_{\alpha\beta\mu\nu\zeta\xi}w^*_\alpha(0|k)\times[D_\nu(k)w_\mu(z|k)]^*_{z=0}[D_\beta(k)D_\xi(k)w_\zeta(z|k)]_{z=0}/k^3, \quad (4.40)$$

which are independent of k. Using Eq. (4.14), we transform this generalized evolution equation into real space to obtain

$$U_\tau+\frac{\partial}{\partial\xi}\{\beta_1\hat{H}U_\xi+\beta_2U_{\xi\xi}+\alpha_1U_\xi\hat{H}U+\alpha_2U\hat{H}U_\xi+\alpha_3\hat{H}(UU_\xi)\}=0. \quad (4.41)$$

The real parameters $\alpha_1,\alpha_2,\alpha_3$ are linear combinations of the real coefficients $\kappa_1,\kappa_2,\kappa_3$. Equation (4.41) has conservation form with the most general nonlocal nonlinear flux of second order that involves one Hilbert transform and one spatial derivative. In addition, the lowest-order dispersion term is nonlocal too, and of the Benjamin-Ono type. The evolution equations (4.15) and (4.19) governing nonlinear Rayleigh wave propagation in a linear substrate covered by a nonlinear film may be regarded as special cases of this class of nonlocal evolution equations.

V. SOLITARY SOLUTIONS

In the following, we are specifically interested in solitary wave and stationary periodic wave solutions in the case $\beta_2=0$. A traveling wave ansatz in Eq. (4.41) with $\eta=\xi-v\tau$ leads immediately to

$$\widetilde{U}-\beta_1\hat{H}\widetilde{U}_\eta-\beta_2\widetilde{U}_{\eta\eta}-\alpha_1\widetilde{U}_\eta\hat{H}\widetilde{U}-\alpha_2\widetilde{U}\hat{H}\widetilde{U}_\eta-\alpha_3\hat{H}(\widetilde{U}\widetilde{U}_\eta)=\text{const}, \quad (5.1)$$

where the parameters have been rescaled by the factor $1/v$. For solitary wave solutions, that we are primarily interested in here, the constant on the right-hand side obviously has to vanish. With the help of the convolution theorem (A3), this may be rewritten in the form

$$\hat{H}\widetilde{U}+\beta_1\widetilde{U}_\eta+\beta_2\hat{H}\widetilde{U}_{\eta\eta}+\frac{\partial}{\partial\eta}\left[\frac{\alpha_1+\alpha_3}{2}\widetilde{U}^2-\frac{\alpha_1}{2}(\hat{H}\widetilde{U})^2\right]+(\alpha_1-\alpha_2)\hat{H}(\widetilde{U}\hat{H}\widetilde{U}_\eta)=0. \quad (5.2)$$

In the transition (5.1) to (5.2), care has to be taken in the presence of periodic solutions with nonvanishing averages of $\widetilde{U}$ and $\widetilde{U}\hat{H}\widetilde{U}_\eta$. Often, the Fourier transform of Eqs. (4.41) and (5.2) is more convenient for further analysis and explicit calculations. Using the definition

$$\tilde{U}(\eta)=\int_{-\infty}^{\infty}dk\;U_k\,\exp(ik\eta) \tag{5.3}$$

with

$$U_{-k}=U_k^*, \tag{5.4}$$

we can rewrite Eq. (5.2) in k space, using property (A9) of the Hilbert transform,

$$0=(1+\beta_1 k+\beta_2 k^2)U_k+\int_0^k dq\;k\frac{1}{2}[\alpha_1+\alpha_2+\alpha_3]U_qU_{k-q}$$
$$+\int_k^{\infty}dq\{k[\alpha_1-\alpha_2+\alpha_3]+2q[\alpha_2-\alpha_1]\}U_qU_{q-k}^* \tag{5.5}$$

for wave numbers $k\geqslant 0$. [The corresponding equation for $k<0$ follows from Eq. (5.5) with Eq. (5.4).] The first of the two nonlinear terms in Eq. (5.5) represents summation processes (k being the sum of the wave numbers q and $k-q$ of the two Fourier amplitudes), including second-harmonic generation. The second term corresponds to difference processes (k being the difference of the wave numbers q and $q-k$ of the two Fourier amplitudes). The structure of Eqs. (5.1) and (5.2) suggests that solutions can be found as even functions of η. Consequently, the Fourier amplitudes U_k may be taken to be real.

A. Nonlinear Rayleigh waves

The integrodifferential equation (3.12), governing traveling nonlinear Rayleigh waves in a linear substrate covered by a nonlinear film, corresponds to the special case $\alpha_1=\alpha_2=0$. If, in addition, $\beta_2=0$, there is the same number of derivatives in front of the linear dispersion term and the nonlinear term. Alternatively, in the Fourier space version (5.5), the linear dispersion term and the nonlinear terms contain the same power of k. As a consequence, the linear dispersion term is not sufficiently efficient in suppressing higher harmonics, and the existence of stationary solutions is not expected.

In a search for periodic stationary solutions, one may proceed as in Ref. [5] and convert Eq. (5.5) into an infinite set of algebraic equations using the ansatz

$$U_k=\delta(k-nq_0)\frac{2}{q_0\alpha_3}Q_n, \tag{5.6}$$

with some fundamental wave number $q_0=2\pi/\lambda$ and $n=1,2,\ldots,\;Q_{-n}=Q_n$. In numerical calculations, the system of coupled algebraic equations is truncated, requiring $Q_n=0$ for $n>N$ with some given integer N. The resulting finite system of equations is solved by a Newton-Raphson routine, and N is then successively increased. Figure 3 shows a resulting Fourier spectrum. Even with $N=400$, there is no indication of a convergence of the Fourier amplitudes Q_n to zero.

When the Benjamin-Ono type dispersion term is replaced by the KdV-type dispersion, i.e., $\beta_1=0,\;\beta_2\neq 0$, periodic stationary solutions have been found numerically with the help of the procedure descibed above. Here, the Fourier amplitudes rapidly converge to zero (Fig. 3). For $\beta_2q_0^2\to 0$, these solutions become periodic pulse trains consisting of strongly localized pulses. Two examples are shown in Fig. 4. A linear stability analysis has been carried out for these periodic solutions on the basis of the evolution equation (4.41). They have been found to exhibit oscillatory instabilities for the range of parameter βq_0^2 investigated [37].

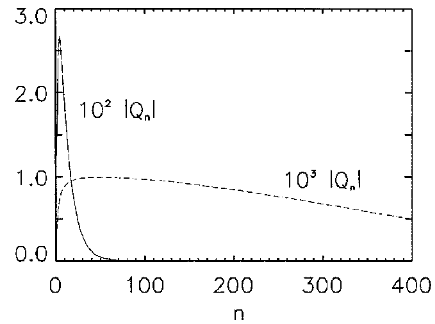

FIG. 3. Moduli of Fourier amplitudes Q_n resulting in the procedure for the search of stationary periodic solutions of Eq. (4.41) using Eqs. (5.5) and (5.6), $\alpha_1=\alpha_2=0$. $\beta_1q_0=0.5$, $\beta_2q_0^2=0$ (dashed line), $\beta_1q_0=0$, $\beta_2q_0^2=0.05$ (solid line).

B. Solitary waves of Benjamin-Ono type

In spite of the fact that for $\beta_2=0$, the linear dispersion term and at least part of the nonlinearity in evolution equation (4.41) carry the same number of spatial derivatives, Eq. (4.41) does have solitary wave solutions for $\beta_2=0$ and certain choices of parameters $\alpha_1,\alpha_2,\alpha_3$, that can even be expressed analytically. The nonlinear term in Eq. (5.5) associated with summation processes is of special form in these cases.

First we shall investigate the case $\alpha_1=-\alpha_2=1,\alpha_3=0,\beta_1=1$, for which our equation has solutions of Benjamin-Ono type. Equation (5.1) then has the form

$$\tilde{U}-\hat{H}\tilde{U}_\eta-\tilde{U}_\eta\hat{H}\tilde{U}+\tilde{U}\hat{H}\tilde{U}_\eta=0. \tag{5.7}$$

Although Eq. (5.7) differs from the corresponding reduction (3.13) of the Benjamin-Ono equation, it admits the same

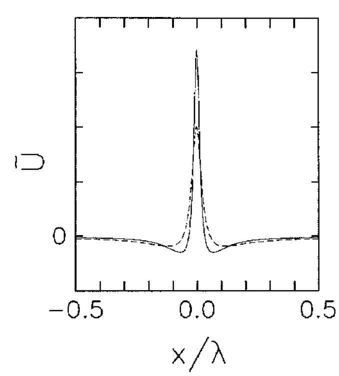

FIG. 4. Periodic pulse train solutions of Eq. (4.41) with $\beta_1=0$, $\alpha_1=\alpha_2=0$, determined via Eqs. (5.5) and (5.6). $\beta_2q_0^2=0.05$ (solid line), $\beta_2q_0^2=0.15$ (dashed line).

solitonlike solution. This solution has powerlike asymptotic behavior as common for solitary pulses in multidimensional systems. The Fourier transform of this solitary solution has a simple form,

$$U_k=\exp(-|k|). \tag{5.8}$$

But in spite of their identical form, the physical nature of these solutions differs. When transforming back to the initial variables, our solution has the form

$$U(\xi,\tau)=\frac{A}{1+B(\xi-v\tau)^2v^2}, \tag{5.9}$$

where A and B are fixed constants for given $\beta_1,\alpha_1,\alpha_2,\alpha_3$. The corresponding solution of the Benjamin-Ono equation differs from Eq. (5.9),

$$U(\xi,\tau)=\frac{\tilde{A}v}{1+\tilde{B}(\xi-v\tau)^2v^2}. \tag{5.10}$$

So, the Benjamin-Ono soliton solution (5.10) contains the familiar relation between the amplitude and the width of the soliton: $U_0\sim 1/\Delta$, where $U_0\sim v$ is the amplitude of a soliton and $\Delta\sim 1/v$ is its width. As in the Benjamin-Ono equation, the solitary wave solution (5.9) has the width $\Delta\sim 1/v$ and this width tends to infinity in the limit $v\rightarrow 0$. But in contrast to the Benjamin-Ono equation the amplitude of this solitary pulse is fixed, which is quite unusual in soliton theory.

By inserting Eq. (5.8) into Eq. (5.5), one may verify that the nonlinear term associated with summation processes vanishes and need not be compensated by a linear dispersion term. Another interesting feature of the choice of parameters $\alpha_1=-\alpha_2,\alpha_3=0$ is that there is no second-harmonic generation in this case.

The solution (5.9) is unphysical for the following reasons. First of all, the longitudinal displacement associated with it has the following form:

$$u_x(x,0,t)=\frac{A}{\sqrt{B}(V-C_R)}\arctan[\sqrt{B}(V-C_R)(x-Vt)] \tag{5.11}$$

and the total deformation of the film $u_x(+\infty,0,t)-u_x(-\infty,0,t)$ would be nonzero. Usually in one-dimensional elastic nonlinear systems the total deformation connected with a soliton is nonzero too [38]. However, in our effectively two-dimensional system only spatial regions near the surface can be deformed and the strain decreases in the depth of the bulk. A nonzero total deformation at the surface would also imply a nonzero deformation $u_x(+\infty,z,t)-u_x(-\infty,z,t)$ independent of the depth z.

Due to the complicated two-dimensional strain distribution in the substrate and due to the connection between the two components of the displacements in the sagittal plane, the z component of the displacement field at the surface would diverge at large distances from the center of the solitary pulse. From Eqs. (B2) and (A4) it follows that

$$u_z|_s\sim\ln(1+\eta^2). \tag{5.12}$$

Following Ono [39] we can find a periodical generalization of the solitary solution (5.9). Using the formulas (A6) it is easy to verify that this solution has the form

$$\tilde{U}(\eta)=\frac{1-b^2}{1-b\cos(\eta/l)}, \tag{5.13}$$

where the period of the wave is $L=2\pi l=2\pi b^2/\sqrt{1-b^2}$ and $b\leqslant 1$. The periodic solution (5.13) transforms into the solitary pulse (5.9) in the limit $b\rightarrow 1$. This periodic solution as well as the solitary one (5.9) would be accompanied by a nonzero average deformation of the surface.

C. Solitary waves in the absence of summation processes

The second combination of parameters α_i, $i=1,2,3$, which leads to simple analytic solutions, is the following:

$$\alpha_1=\alpha_2,\quad \alpha_3=-2\alpha_1. \tag{5.14}$$

We also put $\alpha_1=1,\beta_1=1$. In this case Eq. (5.2) reduces to

$$\hat{H}\tilde{U}+\tilde{U}_\eta-\frac{1}{2}\frac{\partial}{\partial\eta}[\tilde{U}^2+(\hat{H}\tilde{U})^2]=0. \tag{5.15}$$

The same nonlinearity as appearing in Eq. (5.15) had been considered by Hunter [8] in an evolution equation without a linear dispersion term. Hence, the linear parts of Eq. (5.15) and the corresponding evolution equation in Ref. [8] are quite different.

Since the linear dispersion term and the nonlinear term in Eq. (5.15) both appear as a first derivative with respect to η, one would naively expect that no solitary wave solution can exist since the linear dispersion is not strong enough to suppress higher harmonics generated by the nonlinearity. However, inspection of the Fourier-space version of Eq. (5.15),

$$(1+k)U_k=2k\int_k^\infty dq\,U_qU^*_{q-k} \tag{5.16}$$

for $k>0$ reveals that for this special choice of nonlinearity, summation processes are absent. Consequently, generation of higher harmonics does not take place and the existence of solitary waves for this equation does not come as a surprise.

Equation (5.15) has the following solitary wave solution:

$$\tilde{U}(\eta)=4\left[\frac{2}{(\eta^2+1)^2}-\frac{1}{(\eta^2+1)}\right], \tag{5.17}$$

which can be verified by substituting the Fourier transform of the solution (5.17),

$$U_k=2|k|\exp(-|k|), \tag{5.18}$$

into the integral equation (5.16). [Compare this Fourier transform with Eq. (5.8) for the Benjamin-Ono-type solitary pulse.]

The profile of the soliton solution (5.17) is shown in Fig. 5. In contrast to the Benjamin-Ono-type solution of the pre-

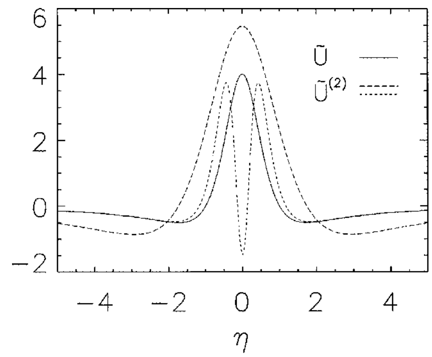

FIG. 5. Solitary wave solutions (5.17) (solid line) and (5.39) with parameters (5.35) (long dashed line) and (5.36) (short dashed line).

ceding subsection, it satisfies the condition $\int_{-\infty}^{\infty}\tilde{U}(\eta)d\eta=0$, that has to be imposed on the strain $\partial u_x/\partial x$ associated with surface waves.

Using the relations (B2), one is led to

$$u_x\sim\frac{1}{V-C_r}\frac{\eta}{1+\eta^2},\quad u_z\sim\frac{1}{\sqrt{V-C_r}}\frac{1}{1+\eta^2}.\tag{5.19}$$

So in contrast to a "usual" soliton the amplitude of these Rayleigh solitary waves diverges in the limit $V\to C_r$.

Using the Green's function of the Laplace equation (see, for example, Ref. [41]) and the relation (1.4) we can easily connect the longitudinal and transverse distributions of deformation on the surface $\partial u/\partial x|_s$ and in the bulk of the substrate $\partial u/\partial x|_v$,

$$\left.\frac{\partial u}{\partial x}\right|_v=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dx'(x-x')}{(x-x')^2+z^2\kappa^2}\hat{H}\left.\frac{\partial u(x')}{\partial x'}\right|_s,\tag{5.20}$$

where $\kappa=\sqrt{1-V^2/C^2}$, $V\simeq C_r$, and $C=C_t$ for transverse and $C=C_l$ for longitudinal deformations.

Substituting the solitary solution (5.17) into the formula (5.20) and rescaling, we obtain the distribution of deformation in the substrate

$$\tilde{U}(\eta,z)=4\left[\frac{2(z+1)^2}{[(z+1)^2+\eta^2]^2}-\frac{1}{(z+1)^2+\eta^2}\right]$$
$$=4\frac{(z+1)^2-\eta^2}{[(z+1)^2+\eta^2]^2},\tag{5.21}$$

where $\tilde{U}(\eta,z)\propto u^l_{x,x}(x,z,t)$. We note that this two-dimensional solution is very close to the lump soliton solution of the Kadomtsev-Petviashvili equation [40],

$$\tilde{U}\sim\frac{z^2+1-\eta^2}{(z^2+1+\eta^2)^2}.\tag{5.22}$$

This fact confirms the two-dimensional nature of the Rayleigh solitary waves. The solitary solution (5.21) can be re-

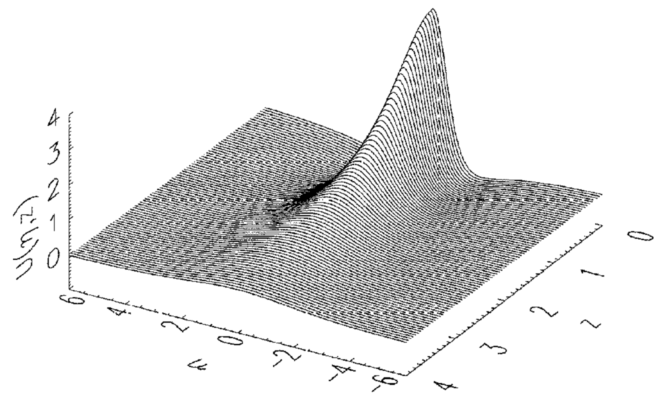

FIG. 6. Two-dimensional representation of the surface acoustic solitary wave (5.21). $\eta\propto x-Vt$, x coordinate parallel, z coordinate normal to the surface.

written in polar coordinates $x=r\sin\phi, z=-1+r\cos\phi$, by which it assumes the simple representation

$$\tilde{U}=\frac{4\cos 2\phi}{r^2}.\tag{5.23}$$

A graphical representation is given in Fig. 6.

By the analogy to the above results (5.9),(5.13) the structure of the periodic generalization of the solitary solution (5.17) is almost evident. Such solution has the form

$$\tilde{U}=\frac{2\gamma^2}{1-\gamma^2}\left[\frac{\gamma^2}{(1-\sqrt{1-\gamma^2}\cos\gamma\eta)^2}-\frac{1}{1-\sqrt{1-\gamma^2}\cos\gamma\eta}\right],\tag{5.24}$$

where the parameter γ ($\gamma\leqslant 1$) characterizes the period of this nonlinear wave, $\gamma=2\pi/L$. In the limit $\gamma\to 0$ the expression (5.24) reduces to the solitary solution (5.17).

In k space the periodic solution (5.24) has a simple form similar to that of the solitary solution (5.17). For periodic solutions with the spatial period L (or with the fundamental wave number $\gamma=2\pi/L$) we introduce Fourier amplitudes Q_n, $n=1,2,\ldots$ via

$$\tilde{U}(\eta)=\gamma\sum_{n=1}^{\infty}Q_n\exp(in\gamma\eta)+\text{c.c.}\tag{5.25}$$

Then we have the following discrete system instead of integral equation (5.16):

$$(1+n\gamma)Q_n=2\gamma^2 n\sum_{m=1}^{\infty}Q_mQ_{m+n},\quad n>0.\tag{5.26}$$

It is easy to obtain the following solution of this equation:

$$Q_n=\frac{2\gamma n}{1-\gamma^2}\exp\left(-n\ln\sqrt{\frac{1+\gamma}{1-\gamma}}\right),\quad n>0.\tag{5.27}$$

Note that this solution has zero average consistent with the exclusion of the $(n=0)$-component in Eq. (5.26). In the limit $\gamma\to 0$, $\gamma n=k$ this solution transforms into the solitary solution (5.17).

An interesting property of Eq. (5.15) is the existence of additional solitary wave solutions which may even form an infinite set of solutions. Such solutions can be sought by using the following ansatz in Eq. (5.16):

$$U_k=\sum_{n=1}^{N} a_n|k|^n e^{-b|k|}, \tag{5.28}$$

which is a generalization of Eq. (5.18), where a_n, $n=1,\ldots,N$ and b are constants to be determined. Inserting this ansatz into Eq. (5.16) and putting all coefficients in front of the different powers of k equal to zero, we get $(N+1)$ algebraic equations for a_n, $n=1,\ldots,N$ and b, i.e., we have as many equations as we have parameters to determine. These equations are

$$a_1=2\int_0^\infty dq\ e^{-2bq}f^2(q), \tag{5.29}$$

$$a_s+a_{s+1}=\frac{2}{s!}\int_0^\infty dq\ e^{-2bq}f(q)f^{(s)}(q),\quad s=1,\ldots,N-1, \tag{5.30}$$

$$a_N=\frac{2}{N!}\int_0^\infty dq\ e^{-2bq}f(q)f^{(N)}(q), \tag{5.31}$$

where $f(q)$ is the polynomial,

$$f(q)=\sum_{n=1}^{N} a_n q^n \tag{5.32}$$

and $f^{(s)}(q)=\partial^s f(q)/\partial q^s$. For example, there are three equations for a_1,a_2,b in the case of $N=2$,

$$\begin{aligned} a_1&=a_1a_2 2I_3+a_2^2 2I_4,\\ 1&=a_1 2I_2+a_2 2I_3,\\ 1&=a_1I_1+a_2I_2, \end{aligned} \tag{5.33}$$

where

$$I_n(b)=2\int_0^\infty dq\ e^{-2bq}q^n=\frac{2n!}{(2b)^{n+1}}. \tag{5.34}$$

In addition to the previous solution (5.18) with $a_1=2, a_2=0, b=1$ the system of equations (5.33) has two additional solutions

$$a_1=\frac{3}{2}(3+\sqrt{3}),\quad a_2=\frac{3}{2}(3+2\sqrt{3}),\quad b=\frac{1}{2}(3+\sqrt{3}) \tag{5.35}$$

and

$$a_1=\frac{3}{2}(3-\sqrt{3}),\quad a_2=\frac{3}{2}(3-2\sqrt{3}),\quad b=\frac{1}{2}(3-\sqrt{3}). \tag{5.36}$$

For the calculation of solitary solutions in coordinate space we can use the formula

$$\int_0^\infty dk\cos k\eta k^n e^{-b|k|}=\left(-\hat{H}\frac{d}{d\eta}\right)^n\frac{b}{b^2+\eta^2}, \tag{5.37}$$

and represent the solution of order N in the form

$$\tilde{U}^{(N)}(\eta)=2\sum_{n=1}^{N} a_n\left(-\hat{H}\frac{d}{d\eta}\right)^n\frac{b}{b^2+\eta^2}. \tag{5.38}$$

It is evident from this expression that $\int_{-\infty}^{\infty}\tilde{U}(\eta)d\eta=0$. In the particular case of the above solutions (5.35) and (5.36) we obtain

$$\tilde{U}^{(2)}(\eta)=2a_1\left(\frac{2b^2}{(\eta^2+b^2)^2}-\frac{1}{(\eta^2+b^2)}\right)+2a_2\left(\frac{8b^3}{(\eta^2+b^2)^3}-\frac{6b}{(\eta^2+b^2)^2}\right). \tag{5.39}$$

The profiles of these solitary solutions are shown in Fig. 5. We see that the field distributions in the solution (5.17) and the solution with parameters (5.35) are very similar. But the displacement profiles in the substrate are quite different. It is easy to show that after using formula (5.20) with the result (5.38) we can rewrite the two-dimensional solutions for the Rayleigh solitary waves as

$$\begin{aligned}\tilde{U}(\eta,z)=&\left[-a_1\frac{\partial}{\partial\eta}+a_3\frac{\partial^3}{\partial\eta^3}-a_5\frac{\partial^5}{\partial\eta^5}+\cdots\right]\\ &\times\frac{1}{\pi}\int_{-\infty}^{\infty}d\eta' F(\eta'-\eta,z)\frac{b}{b^2+(\eta')^2}\\ &+\left[-a_2\frac{\partial^2}{\partial\eta^2}+\alpha_4\frac{\partial^4}{\partial\eta^4}-a_6\frac{\partial^6}{\partial\eta^6}-\cdots\right]\\ &\times\frac{1}{\pi}\int_{-\infty}^{\infty}d\eta' F(\eta'-\eta,z)\frac{\eta'}{b^2+(\eta')^2},\end{aligned} \tag{5.40}$$

where $F(\eta'-\eta,z)$ is the kernel of the integral in Eq. (5.20),

$$F(\eta'-\eta,z)=\frac{1}{(\eta'-\eta)-iz}+\frac{1}{(\eta'-\eta)+iz}. \tag{5.41}$$

It is convenient to introduce polar coordinates with origin having distance b from the surface of the half space outside the substrate,

$$z=-b+r\cos\phi,\quad \eta=r\sin\phi. \tag{5.42}$$

In these coordinates the expressions for the solitary waves in the general case are much simpler,

$$\tilde{U}^{(N)}(\eta,z)=2\sum_{n=1}^{N} a_n \frac{\cos(n+1)\phi}{r^{n+1}}. \tag{5.43}$$

Consequently, in two-dimensional space the higher-order solitary solutions have different symmetry.

For $N=3$ there exist two additional solutions. The first having parameters $b\simeq 4.07$, $a_1\simeq 15.48$, $a_2\simeq 47.49$, $a_3\simeq 131.19$ has the profile similar to the expression (5.39) with the parameters (5.35), i.e., has only one zero; the second solution with $b\simeq 0.15$, $a_1\simeq -0.05$, $a_2\simeq 0.04$, $a_3\simeq -0.006$ has the symmetry of the previous solution (5.39) with parameters (5.36).

CONCLUSIONS

The main goal of this paper has been an investigation into existence and properties of solitary surface acoustic waves propagating in a homogeneous elastic half space with linear dispersion and nonlinearity introduced via the surface. For this purpose, we have derived effective boundary conditions at the surface of the elastic half space that represent a strongly nonlinear thin film. An advantage of surface acoustic waves over other wave systems is the possibility of easily manipulating their propagation properties via the surface. Especially linear dispersion can be tailored by coating the surface or by letting other materials diffuse into or react with the substrate material. For the latter reason, we have also considered half spaces with continuously varying material properties and strong nonlinearity near the surface. For these systems, evolution equations for nonlinear Rayleigh waves have been derived that contain nonlocal linear dispersion and nonlocal second-order nonlinearity.

Subsequently, the effective boundary conditions representing a nonlinear film have been generalized. For this generalized system, an evolution equation has been derived that contains three nonlocal nonlinear terms. The physically more relevant case of a nonlinear film covering a linear substrate forms a special case. For this case, numerical analysis revealed that solitary wave solutions are likely to exist only in the presence of higher-order linear dispersion. Periodic pulse train solutions have been computed that have been found unstable, exhibiting oscillatory instabilities.

Analytic solitary wave solutions have been found for two other special cases. In one of these, a whole family of traveling solitary wave solutions has been identified, its members having different shapes. These solitary pulses exhibit an algebraic decay both into the substrate and along the surface.

For the evolution equations with nonlocal nonlinearity as studied in this paper, analysis in Fourier space often proves to be preferable to real space. On the one hand, solutions can be found more easily in this way. On the other hand, decomposition of the nonlinearity in Fourier space into summation and difference processes and inspection of these two parts yields a better understanding of the counterplay of linear dispersion and nonlinearity in nonlocal evolution equations.

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APPENDIX A

We use the following definition of the Hilbert transform Ref. [42]:

$$\hat{H}f(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dx'}{x'-x}f(x'). \tag{A1}$$

In addition to the well-known skew-symmetric relationship

$$\hat{H}\hat{H}f(x)=-f(x), \tag{A2}$$

the Hilbert transform satisfies the convolution theorem Ref. [43],

$$\hat{H}(f\varphi)=f\hat{H}\varphi+\varphi\hat{H}f+\hat{H}[(\hat{H}f)(\hat{H}\varphi)]. \tag{A3}$$

For the study of solitary waves the following formulas are useful [42]:

$$\hat{H}(1/R)=-x/aR,\quad \hat{H}(1/R^n)=\frac{1}{2(n-1)a}\frac{\partial}{\partial a}\hat{H}\frac{1}{R^{n-1}}, \tag{A4}$$

$$\hat{H}(x/R)=a/R,\quad \hat{H}(x/R^n)=-\frac{1}{2(n-1)a}\frac{\partial}{\partial a}\hat{H}\frac{x}{R^{n-1}}, \tag{A5}$$

where $R=a^2+x^2$ and a is a parameter.

For the investigation of periodic waves one may use [39]

$$\hat{H}\frac{1}{1-b\cos ax}=-\operatorname{sgn} a\left(\frac{b\sin ax}{\sqrt{1-b^2}(1-b\cos ax)}\right), \tag{A6}$$

$$\hat{H}\frac{\sin ax}{1-b\cos ax}=-\operatorname{sgn} a\left(\frac{\sqrt{1-b^2}}{b(1-b\cos ax)}-\frac{1}{b}\right). \tag{A7}$$

Finally, for the Fourier transformation of the equations and solutions in this paper we use

$$\hat{H}\sin kx=\frac{k}{|k|}\cos kx, \tag{A8}$$

$$\hat{H}\exp(ikx)=i\frac{k}{|k|}\exp(ikx). \tag{A9}$$

APPENDIX B

The connection between the components of deformation associated with a linear monochromatic wave at the planar surface of an elastic half space is

$$\frac{\partial u_x^l}{\partial x}=v, \quad \frac{\partial u_x^t}{\partial x}=-\sqrt{1-V^2/C_t^2}\hat{H}w,$$

$$\frac{\partial u_x^l}{\partial z}=\sqrt{1-V^2/C_l^2}\hat{H}v, \quad \frac{\partial u_x^t}{\partial z}=(1-V^2/C_t^2)w,$$

$$\frac{\partial u_z^l}{\partial x}=\sqrt{1-V^2/C_l^2}\hat{H}v, \quad \frac{\partial u_z^t}{\partial x}=w,$$

$$\frac{\partial u_z^l}{\partial z}=-(1-V^2/C_l^2)v, \quad \frac{\partial u_z^t}{\partial z}=\sqrt{1-V^2/C_t^2}\hat{H}w. \tag{B1}$$

It is then easy to find the expressions for the total deformations using the relations (B1) and (1.6),

$$\frac{\partial u_x}{\partial x}=\frac{V^2}{2C_t^2}v, \quad \frac{\partial u_z}{\partial z}=-\left(1-2\frac{C_t^2}{C_l^2}\right)\frac{V^2}{2C_t^2}v,$$

$$\frac{\partial u_z}{\partial x}=-\sqrt{\frac{\kappa_l}{\kappa_t}}\frac{V^2}{2C_t^2}\hat{H}v, \quad \frac{\partial u_x}{\partial z}=\left(\kappa_l-\kappa_t+\frac{V^2}{2C_t^2}\right)\hat{H}v. \tag{B2}$$

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